臺灣大學數學系

八十八學年度第一學期碩博士班資格考試試題

微分方程式

* 以下七題選四題

1. $$\text{Solve} \begin{cases} u_x + (1+y)u_y = u^2 \\ u(x,0) = h(x) \end{cases}$$

2. Solve the initial-boundary-value problem

$$\begin{array}{lll} u_{tt} = u_{xx} & for & 0 < x < \pi, 0 < t, \\ u = 0 & for & x = 0, t > 0, \\ u = 0 & for & x = \pi, t > 0, \\ u = 1 & for & 0 < x < \pi, t = 0. \end{array}$$

3. Suppose u is harmonic on Ω, and the ball $\{x : |x - \xi| \leq \rho\} \subset \Omega$. Show that

$$|Du(\xi)|^2 \leq \frac{1}{\omega_n \rho^{n-1}} \int_{|x-\xi|=\rho} |Du(x)|^2 dS_x,$$

   where $\omega_n \rho^{n-1}$ is the surface area of the sphere $|x - \xi| = \rho$.

4. For $x, y, t \in \mathbb{R}$, $t \neq 0$, define

$$K(x,y,t) = (4\pi|t|)^{-\frac{1}{2}} e^{-(x-y)^2/4t}$$

   Show that for $s > 0, t > 0$,

$$K(x,0,s+t) = \int K(x,y,t)K(y,0,s)dy$$

   holds.

5. Assume $f(x)$ is bounded and continuous on $\mathbb{R}^n$ and satisfies $\int_{\mathbb{R}^n} |f(y)|dy < \infty$. Let u be a bounded solution of

$$\begin{cases} u_t = \triangle u \; for \; x \in \mathbb{R}^n, t > 0 \\ u(x,0) = f(x). \end{cases}$$

   Show that $\lim t \infty u(x,t) = 0$

6. Let $u \in C^2$ and

$$\begin{cases} u_{tt} - \Delta u = 0 \, for \, x \in \mathbb{R}^3, t \geq 0 \\ u(x,0) = f(x), u_t(x,0) = g(x) \end{cases}$$

Suppose f, g have compact support (that is , $f(x) = 0$ and $g(x) = 0$ when $|x|$ is large). Show that there is a constant C such that $|u(x,t)| \leq \frac{C}{t}$ for $x \in \mathbb{R}^3$, $t > 0$

[Hint:

$$u(x,t) = \frac{1}{4\pi t^2} \int_{|y-x|=t} [tg(y) + f(y) + \sum f_{y_i}(y)(y_i - x_i)] dS_y]$$

7.

$$\text{Let} \begin{pmatrix} w_1(x,y,t) \\ w_2(x,y,t) \end{pmatrix}, \text{ and} \begin{pmatrix} u_1(x,y,t) \\ u_2(x,y,t) \end{pmatrix} \text{satisfying}$$

$$\begin{cases} \begin{pmatrix} 5 & 1 \\ 1 & 4 \end{pmatrix} \frac{\partial u}{\partial t} + \begin{pmatrix} 1 & -1 \\ -1 & 0 \end{pmatrix} \frac{\partial u}{\partial x} + \begin{pmatrix} -2 & 0 \\ 0 & 3 \end{pmatrix} \frac{\partial u}{\partial y} = w \quad for \quad t \geq 0 \\ u(x,y,0) = 0 \quad for \quad (x,y) \in \mathbb{R}^3 \end{cases}$$

Suppose u has compact support in (x,y) for $0 \leq t \leq T$. Show that there is a constant C_T such that

$$\int_0^T \int_{\mathbb{R}^2} (u_1^2 + u_2^2) dxdy \; dt \leq C_T \int_0^T \int_{\mathbb{R}^2} (w_1^2 + w_2^2) dxdy \; dt.$$